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Coherent structures in ion temperature gradient turbulence-zonal flow

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Nonlinear stationary structure formation in the coupled ion temperature gradient (ITG)-zonal flow system is investigated. The ITG turbulence is described by a wave-kinetic equation for the action density of the ITG mode, and the longer scale zonal mode is described by a dynamic equation for the $m=n=0$ component of the potential. Two populations of trapped and untrapped drift wave trajectories are shown to exist in a moving frame of reference. This novel effect leads to the formation of nonlinear stationary structures. It is shown that the ITG turbulence can self-consistently sustain coherent, radially propagating modulation envelope structures such as solitons, shocks, and nonlinear wave trains. © 2014 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4898207>]

I. INTRODUCTION

Heat and particle transport in modern day tokamak plasmas is dominantly turbulent. Coherent structures, such as zonal flows are excited self-consistently via nonlinear turbulent Reynolds stresses due to drift wave turbulence, which then back-react (via induced diffusion in \vec{k} space) on the underlying turbulence, resulting in its suppression.^{1–6} Zonal flows are defined as poloidal and toroidally symmetric ($q_\theta = q_z = 0$, where \mathbf{q} is the large scale wavenumber) potential perturbations with a finite radial scale q_r^{-1} significantly larger than the scale of the underlying small scale turbulence, $q_r \ll k_r$, where \mathbf{k} is the wave vector of the small scale turbulence.

A standard renormalized weak turbulence theory à la direct interaction approximation or eddy damped quasi-normal Markovian approximation can be formulated for the drift wave turbulence.⁷ Such an approach assumes:

1. The variation in phase space velocities $\Delta(\Omega/q)$ of the modulation “wave” (Ω and q correspond to frequency and radial wave number of modulation) is smaller than the variation in the velocity of the quasi-particle $(\partial v_g / \partial k) \Delta k$, Δk being width of island in the phase for the quasi-particles orbit, so that resonant island overlap can occur. That is Chirikov parameter $S = (\partial v_g / \partial k) \Delta k / \Delta(\Omega/q) > 1$.
2. The amplitude of modulation is sufficiently low so that the quasi-particle trajectories are unperturbed. Alternatively, this amounts to saying that the Kubo number $\mathcal{K} \equiv \omega_b / \gamma$ is small (i.e., $\mathcal{K} < 1$), where ω_b is the bounce frequency of the quasi-particle and $1/\gamma$ is its life time.

Such a renormalized weak turbulence theory is able to explain certain features of the observed turbulence. However, numerical simulations and experiments show that drift wave turbulence exhibits several non-Gaussian features like intermittency, bursty transport, etc.^{8–20} These characteristics are

related to the presence of coherent structures in drift-wave turbulence^{21–23} and cannot be described by renormalized weak turbulence theories. Nonlinear coherent structure formation in the zonal flow envelope in simple drift wave zonal flow system has been studied in Refs. 24–27. This paper aims at describing the physics of nonlinear coherent structures in the coupled ion temperature gradient (ITG) driven turbulence and zonal flows. The methodology used is quite similar to Kaw *et al.*²⁵ and Das *et al.*²⁸ The short wave high frequency drift wave turbulence is treated as gas of quasi-particles, which is described by the wave kinetic equation (WKE)²⁹ for the wave action density N_k , and the long scale zonal flow structures are described by an evolution equation for $m=n=0$ component of the electrostatic potential Φ_q which is basically the flux surface averaged potential vorticity equation. The coupled equation for N_k and Φ_q shows that the trapping of quasi-particles in the effective potential trough generated by the zonal flow profile in strong turbulence regime $\mathcal{K} \gg 1$ but $S \ll 1$. This leads to possibility of coherent structures which can be described in the form of Bernstein-Greene-Kruskal (BGK) waves.³⁰ Reynold stresses offered by the trapped and untrapped ITG waves act in synergy to generate novel coherent structures like solitons, shocks, and nonlinear wave trains in the zonal flow field in the strongly turbulent state.

II. BASIC TURBULENCE EQUATIONS

We start with the nonlinear fluid equations for ion density perturbation n_i and ion temperature perturbation T describing the background toroidal ITG turbulence³¹

$$\begin{aligned} \frac{\partial n_i}{\partial t} + \frac{\partial \phi}{\partial y} - \varepsilon_n \frac{\partial}{\partial y} (\phi + p_i) - \left(\frac{\partial}{\partial t} - K \frac{\partial}{\partial y} \right) \nabla_\perp^2 \phi \\ + [\phi, n] - [\phi + p, \nabla_\perp^2 \phi] - \left[\frac{\partial p}{\partial x}, \frac{\partial \phi}{\partial x} \right] - \left[\frac{\partial p}{\partial y}, \frac{\partial \phi}{\partial y} \right] = 0. \end{aligned} \quad (1)$$

Temperature perturbation evolution equation

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$$\left(\frac{\partial}{\partial t} - \frac{5}{3}\tau_i\epsilon_n\frac{\partial}{\partial y}\right)T + \left(\eta_i - \frac{2}{3}\right)\frac{\partial\phi}{\partial y} - \frac{2}{3}\frac{\partial n_i}{\partial t} + \left[\phi, T - \frac{2}{3}n_i\right] = 0. \quad (2)$$

The electron response is assumed to be adiabatic

$$n_e = \phi - \Phi_{ZF}, \quad (3)$$

where Φ_{ZF} is the flux surface averaged zonal potential. The above set of equations are supplemented by the quasi-neutrality condition

$$n_e = n_i. \quad (4)$$

The parallel ion motion is neglected under the assumption of $\omega > k_{\parallel}c_s$. Here, the mixing length normalization is used for the fluctuating quantities like potential perturbation ϕ , density perturbation n_i , pressure perturbation p_i , and temperature perturbation T : $(\phi, n_i, p, T) = (e\delta\phi/T_{0e}, \delta n_i/n_0, \tau_i\delta p_i/P_0, \delta T_i/T_0)L_n/\rho_s$.

The space-time coordinates are normalized as $x = (x - x_0)/\rho_s$, $y = y/\rho_s$, $z = z/L_n$, $t = tc_s/L_n$. And the non-dimensional parameters are $\eta_i = L_n/L_T$, $K = \tau_i(1 + \eta_i)$, $\tau_i = T_{0i}/T_{0e}$, and $\epsilon_n = 2L_n/R$. Here, x , y , z represent orthogonal slab coordinates. L_n and L_T are equilibrium density and temperature scale lengths, R is the major radius of the tokamak, c_s is the ion sound speed at electron temperature, and ρ_s represents the ion sound radius.

Mode energetics: Defining

$$E_\phi = \frac{1}{2} \int d^3x (|\phi|^2 + |\nabla_\perp \phi|^2), \quad (5)$$

and

$$E_T = \frac{3\tau_i}{2(1 + \tau_i)} \frac{1}{2} \int d^3x |T|^2, \quad (6)$$

it is easy to show that

$$\partial_t(E_\phi + E_T) = \frac{3\tau_i}{2(1 + \tau_i)} \eta_i \int d^3x T(-\partial_y \phi), \quad (7)$$

where we have used the identity $\int d^3x f[g] = 0$. This shows that the energy of the system grows as the electrostatic turbulent flux $Q_{ES} = \int d^3x T(-\partial_y \phi)$ draws energy from the mean gradients. Linearizing Eqs. (1) and (2), and using the adiabaticity Eq. (3) and the quasi-neutrality Eq. (4), and eliminating ϕ and T yields the dispersion relation

$$\omega^2(1 + k_\perp^2) - \omega k_y \left(1 - \left(1 + \frac{10\tau_i}{3}\right)\epsilon_n - k_\perp^2 \left(K + \frac{5}{3}\tau_i\epsilon_n\right)\right) + \tau_i\epsilon_n k_y^2 \left(\eta_i - \frac{7}{3} + \frac{5}{3}(1 + \tau_i)\epsilon_n + \frac{5}{3}Kk_y^2\right) = 0. \quad (8)$$

The unstable root $\omega = \omega_r + i\gamma$ from Eq. (8) is

$$\omega_r = \frac{k_y}{2(1 + k_\perp^2)} \left(1 - \left(1 + \frac{10\tau_i}{3}\right)\epsilon_n - k_\perp^2 \left(K + \frac{5}{3}\tau_i\epsilon_n\right)\right), \quad (9)$$

$$\gamma = \frac{|k_y|}{1 + k_\perp^2} \sqrt{\tau_i\epsilon_n(\eta_i - \eta_{th})}, \quad (10)$$

where the threshold η for the instability is given by

$$\eta_{th} \approx \frac{2}{3} - \frac{1}{2\tau} + \frac{1}{4\tau\epsilon_n} + \epsilon_n \left(\frac{1}{4\tau} + \frac{10\tau}{9}\right). \quad (11)$$

While the effect of finite Larmor radius (FLR) has been retained in the real frequency estimation due to its importance for the determination of dispersive effects, it has been neglected in obtaining the linear stability threshold. By taking Fourier transform of Eq. (2), we find that the Fourier amplitudes T_k and ϕ_k are related by

$$T_k = \delta_k \phi_k, \quad (12)$$

where

$$\delta_k = \frac{\left(\eta_i - \frac{2}{3}\right)k_y + \frac{2}{3}\omega_k}{\omega_k + \frac{5}{3}\tau_i\epsilon_n k_y}.$$

By using Eqs. (9) and (10), one can show that δ_k is independent of k up to order k_\perp^2 . In the long wavelength limit $k_\perp^2 < 1$, the real frequency can be expressed as

$$\omega_r = k_y(a - bk_\perp^2), \quad (13)$$

where

$$a = \frac{1}{2} \left[1 - \left(1 + \frac{10\tau_i}{3}\right)\epsilon_n\right], \quad (14)$$

and

$$b = \frac{1}{2} \left[1 + K - \left(1 + \frac{5}{3}\tau_i\right)\epsilon_n\right], \quad (15)$$

and hence the radial group velocity becomes

$$v_{gr} \approx -2k_x k_y b. \quad (16)$$

Defining $W_* = 3\tau_i|\delta_k|^2/(2(1 + \tau_i))$, the mode energy density ε_k can be obtained from Eqs. (5) and (6)

$$\varepsilon_k = (1 + k_\perp^2 + W_*)|\phi_k|^2, \quad (17)$$

and so the wave action density becomes

$$N_k = \frac{1 + k_\perp^2 + W_*}{a - bk_\perp^2} |\phi_k|^2. \quad (18)$$

III. ZONAL MODE EQUATIONS

The zonal flows are $m=0$, $n=0$, or $q_y=q_z=0$ but radial wavenumber $q_x \neq 0$ modes. So the evolution equations for the zonal perturbations can be obtained from the flux surface averaged vorticity equation $\langle \vec{\nabla} \cdot \vec{J} \rangle = 0$ and the flux surface averaged temperature equation³²

$$\partial_t \nabla_x^2 \Phi_{ZF} = -(1 + \tau_i) \langle [\phi, \nabla_\perp^2 \phi] \rangle - \tau_i \langle [T, \nabla_\perp^2 \phi] \rangle + \tau_i \langle [\partial_i \phi, \partial_i T] \rangle, \quad (19)$$

$$\partial_t T_{ZF} = -\langle [\phi, T] \rangle. \quad (20)$$

Since

$$\langle [T, \nabla_\perp^2 \phi] \rangle = -\nabla_x^2 \langle \partial_y T \partial_x \phi \rangle - \langle [\partial_i T, \partial_i \phi] \rangle, \quad (21)$$

and

$$\langle [\phi, \nabla_\perp^2 \phi] \rangle = -\nabla_x^2 \langle \partial_y \phi \partial_x \phi \rangle, \quad (22)$$

and for arbitrary f and g using

$$\langle fg \rangle = \text{Re} \int d\vec{k} f_{\vec{k}} g_{\vec{k}}^*. \quad (23)$$

Equations (19) and (20) can be expressed as

$$\partial_t \nabla_x^2 \Phi_{ZF} = \nabla_x^2 \text{Re} \int d\vec{k} k_y k_x (1 + \tau_i + \delta_k \tau_i) |\phi_k|^2, \quad (24)$$

$$\partial_t T_{ZF} = -\nabla_x \text{Re} \int d\vec{k} i k_y \delta_k |\phi_k|^2. \quad (25)$$

Here, Re stands for the real part. In a coexisting system of ITG turbulence and zonal flow, the modulations of micro-scale fields by mesoscale zonal flows conserve wave action density $N_k = \varepsilon_k / \omega_{r,k}$, where ε_k is the energy density of the k th mode with real frequency $\omega_{r,k}$. The action density has the generic form $N_k = N_k(|\phi_k|^2, |T|^2)$ which by using the linear Fourier amplitude relations can be cast in the form $N_k = N_k(|\phi_k|^2)$. Then the modulated non-linear drivers can be expressed as a function of N_k via $\delta|\phi_k|^2 = C_k \delta N_k$. From Eq. (18), the coefficient C_k can be obtained as

$$C_k = \frac{a - b k_\perp^2}{1 + k_\perp^2 + W_*}. \quad (26)$$

The wave kinetic equation is used to describe the evolution of N_k in the presence of mean flows^{24,33–38}

$$\frac{\partial N_k}{\partial t} + \frac{\partial \omega_{r,k}}{\partial \vec{x}} \cdot \frac{\partial N_k}{\partial \vec{x}} - \frac{\partial \omega_{r,k}}{\partial \vec{x}} \cdot \frac{\partial N_k}{\partial \vec{x}} = \gamma_k N_k - \Delta \omega N_k^2, \quad (27)$$

where $\omega_{r,k}$ and γ_k are, respectively, the real frequency and growth rate of the underlying turbulence in the presence of slowly varying mesoscale zonal fields. The first term on the right hand side is the linear growth and the second term is the eddy damping due to nonlinear interactions. We can find the steady-state turbulence spectrum $\langle N_k \rangle$ by letting the right hand side of Eq. (27) to zero, i.e., $\gamma_k \langle N_k \rangle - \Delta \omega \langle N_k \rangle^2 = 0$. To study the stability of such a steady-state, we make a Chapman-Enskog expansion of N_k ; $N_k = \langle N_k \rangle + \delta N_k$, where $\langle N_k \rangle$ is a slowly varying “mean” wave action density, and δN_k is the coherent perturbation, induced by gradients of $\langle N_k \rangle$ in space \vec{x} and \vec{k} . The wave kinetic equation, linearized for δN_k can be written as

$$\left(\frac{\partial}{\partial t} + \vec{v}_g \cdot \frac{\partial}{\partial \vec{x}} + \gamma_k \right) \delta N_k = \frac{\partial \omega_{r,k}}{\partial \vec{x}} \cdot \frac{\partial \langle N_k \rangle}{\partial \vec{k}} + \delta \gamma_k \langle N_k \rangle. \quad (28)$$

Assuming $\Psi = \Psi_q \exp(-i\Omega t + q_x x)$ where $\Psi = \{\delta N_k, \Phi_{ZF}, T_{ZF}\}$, the wave kinetic equation (28) yields

$$\delta N_{k,q} = R_{k,q} \left[\frac{\partial \omega_{r,k}}{\partial \vec{x}} \cdot \frac{\partial \langle N_k \rangle}{\partial \vec{k}} + \delta \gamma_k \langle N_k \rangle \right], \quad (29)$$

where the propagator $R_{k,q}$ is given by

$$R_{k,q} = \frac{i}{(\Omega_q - \vec{q} \cdot \vec{v}_g + i\gamma_k)}. \quad (30)$$

The zonal flow, being a mesoscale mode, will convect the microturbulence. This effect is captured via $\partial_t \rightarrow \partial_t + [\langle \phi \rangle, \quad]$. We can also write the effect of the zonal perturbations on the dispersion relation

$$\begin{aligned} \delta \omega_{r,k} &= \frac{\delta \omega_r}{\delta \eta_i} \delta \eta_i + \vec{k}_\perp \cdot \langle V \rangle_{E \times B}, \\ &= \frac{k_y k_\perp^2 \tau_i}{2(1 + k_\perp^2)} \nabla_x T_{ZF} + k_y \nabla_x (1 + \tau_i \nabla_x^2) \Phi_{ZF}. \end{aligned} \quad (31)$$

Here, the first term represents the frequency modulation due to modulation in η_i by zonal temperature perturbations T_{ZF} , and the second term represents the frequency modulation due to zonal potential perturbations Φ_{ZF} with FLR corrections. The FLR corrections to the drift wave turbulence equations are sub-dominant in the parameter regime $T_i \ll T_e$, where the wave dispersion is determined by the ion polarization drift effects, and are hence neglected.²⁵ The modulation in growth rate is given by the modulation in η_i by zonal temperature perturbations T_{ZF}

$$\delta \gamma_k = \frac{\partial \gamma_k}{\partial \eta_i} \delta \eta_i = -\frac{|k_y|}{4} \left(\frac{\epsilon_n \tau_i}{\eta_i - \eta_{th}} \right)^{1/2} \nabla_x T_{ZF}. \quad (32)$$

Using Eqs. (31) and (32) yields

$$\delta N_{k,q} = R_{k,q} \left[-k_y \bar{q}^2 \Phi_q \frac{\partial \langle N_k \rangle}{\partial k_x} - i |k_y| W_\eta q T_q \langle N_k \rangle \right], \quad (33)$$

where $\bar{q}^2 = q^2(1 - \tau_i q^2)$ and $W_\eta = \sqrt{\epsilon_n \tau_i / (\eta_i - \eta_{th})} / 4$. Finally, using Eq. (33) the zonal flow Eqs. (24) and (25) takes the form

$$\begin{aligned} \partial_t \Phi_q &= \text{Re} \int d\vec{k} k_y k_x (1 + \tau_i + \delta_k \tau_i) C_k R_{k,q} \\ &\quad \times \left[-k_y \bar{q}^2 \Phi_q \frac{\partial \langle N_k \rangle}{\partial k_x} - i |k_y| W_\eta q T_q \langle N_k \rangle \right], \\ \partial_t T_q &= q \text{Re} \int d\vec{k} k_y \delta_k C_k R_{k,q} \left[-k_y \bar{q}^2 \Phi_q \frac{\partial \langle N_k \rangle}{\partial k_x} - i |k_y| W_\eta q T_q \langle N_k \rangle \right]. \end{aligned} \quad (34)$$

Now using the \vec{k} symmetry properties of δ_k , it is easy to verify that the cross coupling terms survive if and only if $\langle k_x \rangle \neq 0$, where $\langle k_x \rangle = \int d\vec{k} k_x N_k / \int d\vec{k} N_k$ is spectrally averaged k_x . In case $\langle k_x \rangle = 0$, the zonal potential and zonal temperature are excited independently. The independent zonal potential excitation criterion is $-k_x \partial \langle N_k \rangle / \partial k_x > 0$ while any kind of spectrum can excite zonal temperature. Note that

negative exponent of N_k produces negative viscosity at the scale of modulation (q^{-1}), and hence zonal flow generation is also viewed as a negative viscosity phenomenon.^{5,39-44}

IV. COHERENT STRUCTURES: STATIONARY SOLUTIONS

We now look for non-linear coherent stationary structures which are exact solutions of Eqs. (24), (25), and (27) when modulational instability has already saturated. We seek stationary solutions in the absence of source and sink ($\gamma_k = \Delta\omega_k = 0$) in a frame moving with velocity U . Space-time coordinates (ξ, T) in the moving frame are related to the space-time coordinates (x, t) that in the rest frame by the transformation $\xi = x - Ut$ and $T = t$. So $\frac{\partial}{\partial t} = \frac{\partial}{\partial T} - U \frac{\partial}{\partial \xi}$ and $\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}$, $k_x = k_\xi$. Defining zonal flow velocity as $v = \nabla_x \Phi_{ZF}$, we can write stationary equation in the moving frame as

$$(v_{gx} - U) \nabla_\xi N_k - \nabla_\xi \omega_{r,k} \frac{\partial N_k}{\partial k_x} = 0, \quad (36)$$

$$(\mu \nabla_\xi^2 - \nu) v + U \nabla_\xi v = -\nabla_\xi \int d\vec{k} k_y k_x (1 + \tau_i + Re \delta_k \tau_i) C_k N_k. \quad (37)$$

Here, μ is viscous damping coefficient and ν represents neoclassical drag. Equation (36) conserves N_k along the characteristic given by

$$\frac{dk_x}{d\xi} = -\frac{k_y (d/d\xi)(v + \tau_i v'')}{v_{gx} - U}, \quad \frac{dk_y}{d\xi} = 0. \quad (38)$$

Equation (38) can be readily integrated to give the constant of motion W

$$W = -k_y(v + \tau_i v'') + b k_y k_x^2 + k_x U, \quad k_y = k_{y0}. \quad (39)$$

This suggests an exact solution to Eq. (36) in the form

$$N_k(k_x, k_y, x) \equiv N_k(W(k_x, \xi), k_y). \quad (40)$$

W is physically interpreted as frequency of the ITG mode as seen from a frame moving radially with velocity U including the Doppler shift due to zonal flow velocity perturbations. While passing regions of different v and v'' , the k_x of the mode changes in such a way as to keep k_y and Ω constant. Equation (39) can be expressed as

$$\bar{W} = K_x^2 + f(\xi), \quad (41)$$

where $\bar{W} = W/(b k_y)$, $K_x = k_x + \bar{U}$, $f(\xi) = -(\bar{U}/2)^2 - (v + \tau_i v'')/b$, and $\bar{U} = U/(b k_y)$. Since here \vec{k} acts like \vec{v} in the Liouville equation for particle probability distribution function, Eq. (41) can be interpreted as a sum of kinetic (first term) and potential (second term) energies giving the total energy \bar{W} as the constant of motion. Now if profile of the potential function $f(\xi)$ is such that it has minima and maxima (f_m) then a part of drift wave population satisfying $\bar{W} < f_m$ will get trapped around the minimum of the effective potential. Another part satisfying $\bar{W} > f_m$ will remain untrapped. The characteristic ray equations can be written as

$$\frac{dx}{dt} = -2k_x k_y b, \quad (42)$$

$$\frac{dk_x}{dt} = -k_y \frac{d}{dx} (v + \tau_i v''), \quad (43)$$

$$\frac{dk_y}{dt} = 0. \quad (44)$$

Using Eq. (42), the equation of motion for k_x can be written as

$$\frac{d^2 k_x}{dt^2} = 2k_y^2 b \left(\frac{d^2 v}{dx^2} + \tau_i \frac{d^4 v}{dx^4} \right) k_x. \quad (45)$$

If zonal flow is a propagating structure like $v = v(x - Ut)$ then in the moving frame, the above equation becomes

$$\frac{d^2 k_x}{dt^2} = 2k_y^2 b \left(\frac{d^2 v}{d\xi^2} + \tau_i \frac{d^4 v}{d\xi^4} \right) k_x. \quad (46)$$

Equation (46) possesses oscillatory solution when $v'' + \tau_i v'''' < 0$

$$k_x = k_{x0} \cos(\omega_b t), \quad (47)$$

where the quasi-particle bounce frequency ω_b is given by

$$\omega_b = \sqrt{-2k_y^2 b (v'' + \tau_i v''')}. \quad (48)$$

For a monochromatic zonal flow profile $v = v_0 \cos(q\xi)$, the drift wave quasi-particles will get trapped in the crests of zonal flow field. Near the crest the bounce frequency would be given by

$$\omega_b = \sqrt{2k_y^2 b q^2 v_0}. \quad (49)$$

One can also arrive at the same conclusions by looking at the equation of motion in ξ . How trapping happens in the zonal flow crests can be understood as follows. As the ITG quanta rolls down the zonal flow shear layer from its maximum it loses its k_x or $x(\xi)$ —momentum due to restoring force offered by radially inward group velocity. Eventually, k_x^2 becomes zero and the quasi-particle gets reflected, getting trapped in the crest. This can also be conceived from the phase space plot of (ξ, k_x) , shown in Fig. 1, obtained from numerical solutions of Eqs. (42) and (43) for cosine and sine zonal flow profiles which clearly shows trapped and passing trajectories. The trapping of quasiparticle trajectories in the drift wave zonal flow turbulence was also shown in Ref. 45 using the method of decorrelation trajectory.⁴⁶⁻⁴⁸

So once we realized the existence of trapped and untrapped quasi-particles population densities, the solution for stationary Eqs. (36) and (37) can be obtained by solving the self-consistency condition

$$\begin{aligned} & (\mu \nabla_\xi^2 - \nu + U \nabla_\xi) v \\ &= -\frac{1}{4} (1 + \tau_i + Re \delta_k \tau_i) \nabla_\xi \int_{-\infty}^{+\infty} dk_y^2 \left\{ \int_{f_m}^{\infty} d\bar{W} J N_U(\bar{W}, k_y) \right. \\ & \quad \left. + \int_f^{f_m} d\bar{W} J N_T(\bar{W}, k_y) \right\}, \end{aligned} \quad (50)$$

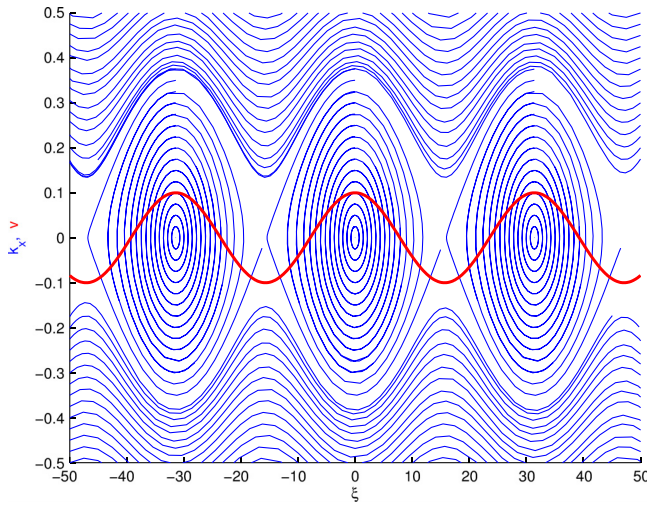


FIG. 1. k_x, v vs ξ when $v = v_0 \cos(q\xi)$ with $\tau = 1$, $\eta_i = 3$, $\epsilon_n = 0.7$, $k_y = 0.2$, $v_0 = 0.1$, and $q = 0.2$.

where N_U and N_T are the action densities for the trapped and untrapped part of the stationary drift wave turbulence, and J is the Jacobian of transformation from k_x to \bar{W} given by

$$J = \frac{(1 - \bar{U}(\bar{W} - f)^{-1/2})(a - bk_y^2 - b((\bar{W} - f)^{1/2} - \bar{U})^2)}{1 + W_* + k_y^2 + ((\bar{W} - f)^{1/2} - \bar{U})^2}. \quad (51)$$

The nonlinearity in Eq. (50) is coming from the dependence of right hand side term on f which in turn is determined by the choice of trapped and untrapped action densities $N_{T,U}$. To figure out the nature of nonlinear structures supported by Eq. (50), we make following choices of N_T and N_U :

$$\frac{N_U}{N_{0U}} = \left[1 + \left(\frac{\bar{W} - f_m}{\Delta} \right)^2 \right]^{-1} \delta(k_y - k_{0y}), \quad \bar{W} > f_m, \quad (52)$$

$$\frac{N_T}{N_{0T}} = \left[1 + \epsilon \left(\frac{f_m - \bar{W}}{\Delta} \right)^{1/2} \right] \delta(k_y - k_{0y}), \quad f < \bar{W} < f_m. \quad (53)$$

The two distributions are chosen to be continuous at $f = f_m$, as shown in Fig. 2. The monochromatic k_y spectrum might appear an extreme idealization, it safely captures the asymmetry in mode propagation direction in poloidal direction. The choice for $N_{U,T}$ is a bit intuitive here. In fact, any function of the constants of motion W, k_y , can be an exact solution of the wave kinetic equation, so there is an arbitrariness in the choice of $N_{U,T}$. If this is so then how to fix $N_{U,T}$? In reality, $N_{U,T}$ should be self-consistently set by the formation of stable nonlinear flow structures at saturation in an initial value problem picture. So for an arbitrary choice of $N_{U,T}$, one should, in principle, check the stability of the corresponding flow structure and iterate the choice of $N_{U,T}$ to finally get a stable structure. Otherwise following Lynden-Bell,⁴⁹ one may ask what is the most probable distribution subject to the constraints imposed by the conservation properties of the WKE and zonal flow equation? This is an important and interesting direction and will be discussed elsewhere. Instead, we made a choice based on intuition.

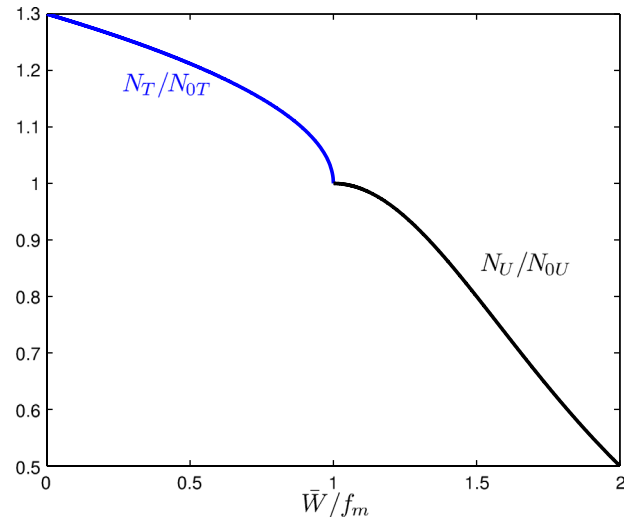


FIG. 2. Population distributions of trapped and untrapped waves for $\epsilon = 0.3$ and $\Delta/f_m = 1.0$.

For example, the two distributions are continuous and have asymptotic power law characteristics. Trapped waves distribution is inspired by the choice of Bohm and Gross⁵⁰ for distribution of trapped particles in an electrostatic wave. It is extremely difficult to evaluate the integrals on the right hand side of Eq. (50) with the Jacobian given by Eq. (51). Hence, some sensible simplifications are desirable. Assuming $k_{\perp}^2 < 1$ and $v_{gx} > U$ and expanding the J up to k_{\perp}^2 allows us to write

$$J = \frac{a}{1 + W_*} \left[1 - \bar{b}k_y^2 - \bar{b}(\bar{W} - f) \right], \quad (54)$$

where $\bar{b} = b/a + 1/(1 + W_*)$. Using this Jacobian, the trapped and untrapped integrals in Eq. (50) are evaluated in the (A) which yields

$$(\mu \nabla_{\xi}^2 - \nu + U \nabla_{\xi})v = -\nabla_{\xi}[\alpha(f_m - f) + \beta(f_m - f)^{3/2} + \alpha_1(f_m - f)^2 + \beta_1(f_m - f)^{5/2}], \quad (55)$$

where $f_m - f = (v - v_{min} + \tau_i(v'' - v'_{min}))/b$ since maximum of f corresponds to minimum of v and

$$\alpha = \bar{a}(N_{0T}\Delta(1 - \bar{b}k_{y0}^2) - N_{0U}\Delta\bar{b}\pi/2)/2, \quad (56)$$

$$\beta = \bar{a}N_{0T}\Delta\epsilon(1 - \bar{b}k_{y0}^2)/3, \quad (57)$$

$$\alpha_1 = \bar{a}N_{0T}\Delta\bar{b}(\Delta/2 - 1)/2, \quad (58)$$

$$\beta_1 = \bar{a}N_{0T}\Delta\epsilon\bar{b}(2\Delta/5 - 2/3). \quad (59)$$

Here, $\bar{a} = (1 + \tau_i + Re\delta_k\tau_i)(ak_{y0})/(1 + W_*)$. Coefficients containing $N_{0T}(N_{0U})$ corresponds to contributions from trapped(untrapped) waves.

Now in the following Eq. (55) is investigated in different limits. First, taking Fourier transform in ξ of the linearized equation gives the dispersion relation $U = -\alpha(1 - \tau_i q^2)/b$. Since α is made of Reynolds stresses both from trapped and untrapped wave contributions, this indicates that coupling to trapped and untrapped waves converts the zonal flow perturbations into radially propagating dispersive waves. Here, no

growth term appears because the resonant waves leading to modulational excitation of zonal flows has already been trapped in the large amplitude zonal flows at saturation. The structure of the nonlinear stationary state is determined by the nonlinear trapped wave contributions, i.e., the terms with β , α_1 , and β_1 in Eq. (55). Neglecting ν and retaining terms up to order 3/2, Eq. (55) becomes

$$V'' + \bar{\mu}V' - \kappa V + \bar{\beta}V^{3/2} = 0, \quad (60)$$

where $V = v - v_{min}$, $\bar{\mu} = \mu/(\alpha\tau_i)$, $\kappa = -(U + \alpha/b)/(\alpha\tau_i)$, and $\bar{\beta} = \beta/(\alpha\tau_i b^{3/2})$. Note that the $V^{3/2}$ nonlinearity arises due to trapped wave population which vanish at low zonal flow amplitudes.

Earlier works on coherent structure of zonal flows in drift wave turbulence reported different other types of nonlinear equations for zonal flows. Tur *et al.*,³⁹ via a multiple space-time scale reductive perturbation method, obtained a sixth order equation with cubic nonlinearity for zonal flows. They showed that no travelling wave solution is supported in zonal flows envelope. Instead, a variety of stationary solutions such as stationary periodic elliptic sinus, stationary kink, and stationary soliton for zonal flow were shown to be possible. Smolyakov *et al.*²⁴ based on higher order (in Ω/qv_{gr}) non-resonant wave action density calculations arrived at a cubic order equation with quadratic nonlinearity in zonal flows. Radially, propagating kink soliton was shown as a solution for the zonal flow. Similarly from perturbative solution of WKE, considering vorticity of zonal flow as a smallness parameter, Itoh *et al.*²⁷ arrived at a quadratic order equation with cubic nonlinearity in zonal flows. They showed a stationary periodic solution for zonal flows. In all these works, the nonlinearity was developed by contributions from untrapped waves only. Hence, origin of $V^{3/2}$ nonlinearity here is fundamentally different from the other zonal flow nonlinearities in the past works just mentioned above.

We now obtain the solutions for Eq. (60) in various limits. For $\bar{\mu} = 0$, Eq. (60) gives the quadrature

$$\frac{1}{2} \left(\frac{dV}{d\xi} \right)^2 + \Psi(V) = \text{const}, \quad (61)$$

where the effective potential is given by

$$\Psi(V) = -\kappa \frac{V^2}{2} + \frac{2}{5} \bar{\beta} V^{5/2}. \quad (62)$$

Such a pseudo-potential will have a minimum if $\kappa > 0$ or $Ub + \alpha < 0$ which is important for coherent structure formation. A schematic of the Sagdeev pseudo-potential is shown in Fig. 3. A pseudo-particle starting at $V > 0$ will oscillate back and forth in the well with a frequency dependent on the amplitude of oscillations. In real space, this situation corresponds to a nonlinear periodic zonal flow V wave train propagating in x with speed U and having spatial period dependent on amplitude. As the initial V approaches zero, the amplitude as well as the period of the nonlinear wave increases. As V starts from zero, the period is infinite and we get a solitary pulse or soliton. In such a case, an exact soliton solution can be written as

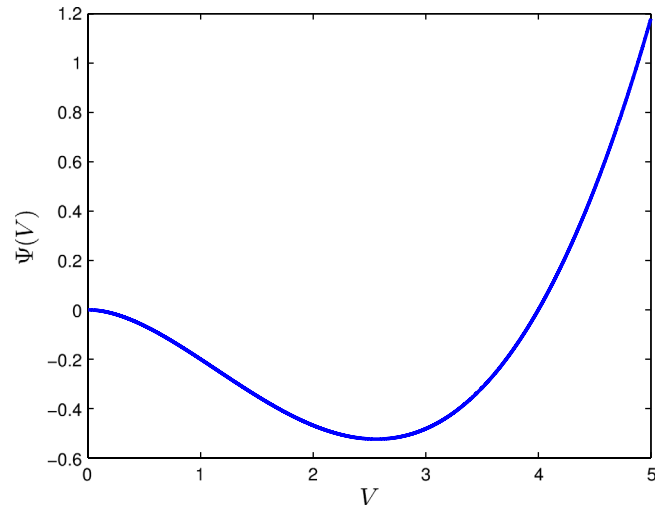


FIG. 3. Sketch of Sagdeev Potential for $\kappa = 0.8$ and $\bar{\beta} = 0.5$.

$$V = \frac{25}{16} \frac{\kappa^2}{\bar{\beta}^2} \sec^4 \left[\frac{\sqrt{\kappa}}{4} (x - Ut) \right]. \quad (63)$$

This soliton structure is a back to back zonal velocity shear layer, as shown in left panel of Fig. 4, with a significant fraction of ITG quasi-particles trapped between them and held together by Reynolds stresses offered by a background population of untrapped quasi-particles and propagating radially in(out), due to coupling between trapped and untrapped quasi-particle populations, when a/b is positive(negative). Now with dissipative terms retained the pseudo-particle suffers a damped oscillation in the pseudo-potential and eventually settles down at the minimum value $V_m = \kappa^4/\bar{\beta}^2$. This corresponds to a shock like structure in zonal flow field $V(\xi)$, see right panel of Fig. 4, with V going from 0 to V_m after oscillating a few times around the final value. For large damping rate, there is no ringing and the shock solution goes monotonically from 0 to V_m . If the dominant dissipation is from viscosity ($\nu = 0$, μ large), then a shock solution can be obtained as

$$V = V_m \left[\frac{\exp(\kappa(x - Ut)/\bar{\mu})}{(1 + \exp(\kappa(x - Ut)/2\bar{\mu}))^2} \right]. \quad (64)$$

Hence, in an initial value scenario the following picture emerges. Beyond a critical gradient, the equilibrium becomes unstable to ITG mode which develops into turbulence. Zonal flow builds up by the nonlinear Reynold stresses with the growth of turbulence intensity. As the zonal flows grow via modulational instability, ITG quasi-particles get trapped in the flows. The trapping causes steepening of the flow. With time the zonal flow amplitude increases, the trapping and so the steepening increases. The moment this steepening becomes large enough to balance the dispersion the zonal flow field takes the shape of a propagating soliton. In presence of dissipative mechanisms like viscosity, such a balance is lost and the end result becomes a propagating oscillatory or monotonic shock structure of the zonal flow.

In order to test these concepts in gyrokinetics, one should use a full-f, global gyrokinetic code which allow

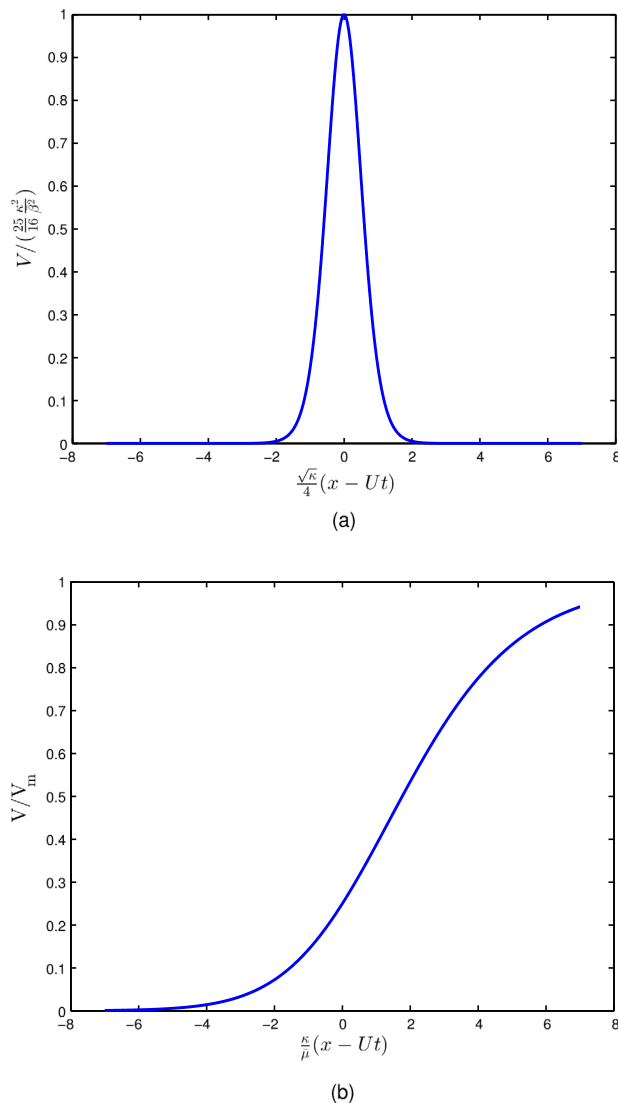


FIG. 4. (a) Zonal flow soliton and (b) zonal flow shock.

proper coupling to large scale flows such as GYSELA,^{51,52} and use two-point, autocorrelation functions $C_x(x + \delta x/2, x - \delta x/2)$, by computing them directly. These can then be studied as Wigner transform $\mathcal{W}(x, k_x) = \int C_x(x + \delta x/2, x - \delta x/2) e^{-ik_x \delta x} d\delta x$ and examined in the phase space consisting of position and wavenumber, where one would look for signatures of wave-trapping in the same way one looks for particle trapping in velocity space. One should trace the time evolution of $\mathcal{W}(x, k_x)$ and zonal flow profile to identify the connection between the flow curvature and wave trapping.

V. CONCLUSIONS

Motivated by the non-Gaussian and intermittent character of observed turbulence, we studied the possibility of coherent zonal structure formation, such as nonlinear wave train, solitons, and shocks, in ITG turbulence in tokamaks in strongly nonlinear regime by looking for stationary solutions of the closed set of equations that describe the mutual interaction of turbulence and zonal flows consisting of the WKE corresponding to the reduced fluid ITG system, coupled with

the equation for the zonal flows, where the Reynolds stress is expressed in terms of the wave action density. These set of equations are similar in structure to the Vlasov-Poisson system of equations where the WKE corresponds to the Vlasov equation and the zonal flows equation correspond to the time derivative of the Poisson equation. Vlasov equation exists in position-momentum phase space of particles while the WKE gives evolution of action density of waves in the position-wavenumber phase space of waves. Stationary solutions of the WKE-Zonal flow system for ITG turbulence are studied in this paper.

It was shown that the results for the drift-wave/zonal flow problem that has been studied within a very similar framework carries over directly into the case of ITG/zonal flow system. In particular, the equations for the trajectories of waves (or quasi-particles), in this framework show that the ITG quasi-particles can get trapped in the “effective potential well” formed by the zonal flow profile in the negative flow curvature regions (i.e., $d^2V/dr^2 < 0$). The trapping happens when $\nu < \gamma_q < \omega_b$ where ν is collisionality, γ_q is the growth rate of the modulational instability, and ω_b is the typical bounce frequency of the ITG quanta in the modulation envelope. Reynolds stress calculations for a population of trapped and untrapped quasi-particles show that the trapped population contributes to $V^{3/2}$ non-linearity in the zonal flow equation, which provides the necessary steepening to balance the dispersion which gives rise to a variety of radially propagating structures such as periodic nonlinear wave-trains, solitons, and shocks in the zonal flow field. These solutions, in fact, represent alternate saturated states of zonal flow generated via the modulational instabilities of the ITG turbulence. In this highly nonlinear regime coherence and quasi-particle trapping are dominant while the usual saturated states (when $\mathcal{K} < 1$ and $S > 1$) are dominated by stochastic quasi-particle motion and diffusion in phase space.⁷

Note that the stationary solutions in this paper were obtained only for zonal potential assuming Bohm-Gross like stationary distribution function for the wave action density in k_x and delta distribution in k_y . It is also assumed that the linear coupling of zonal potential and zonal temperature arising via the nonlinear stresses is weak so that the zonal potential and zonal temperature evolution equations can be effectively decoupled. Improving over these limitations and looking for stationary solutions for zonal temperature, self-consistently and stability of the stationary states are left as a future challenging task.

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APPENDIX: DERIVATION OF EQ. (55)

The trapped and untrapped integrals in Eq. (50) are evaluated first.

Trapped integral (I_T)

$$\begin{aligned}
 I_T &= \int_f^{f_m} d\bar{W} JN_T(\bar{W}, k_y), \\
 &= \int_f^{f_m} d\bar{W} \frac{a}{1+W_*} \left[1 - \bar{b}k_{y0}^2 - \bar{b}(\bar{W} - f) \right] N_{0T} \\
 &\quad \times \left[1 + \epsilon \left(\frac{f_m - \bar{W}}{\Delta} \right)^{1/2} \right]. \quad (A1)
 \end{aligned}$$

Let $\frac{f_m - \bar{W}}{\Delta} = x$ so that $-d\bar{W} = \Delta dx$ then

$$\begin{aligned}
 I_T &= \frac{aN_{0T}\Delta}{1+W_*} \int_0^{f_m-f} dx \left[1 - \bar{b}k_{y0}^2 - \bar{b}(f_m - f - x\Delta) \right] [1 + \epsilon x^{1/2}], \\
 &= \frac{aN_{0T}\Delta}{1+W_*} \int_0^{f_m-f} dx \left[1 - \bar{b}k_{y0}^2 - \bar{b}(f_m - f) \right. \\
 &\quad \left. + \epsilon(1 - \bar{b}k_{y0}^2 - \bar{b}(f_m - f))x^{1/2} + \bar{b}\Delta x + \bar{b}\Delta\epsilon x^{3/2} \right], \\
 &= \frac{aN_{0T}\Delta}{1+W_*} \left[(1 - \bar{b}k_{y0}^2)(f_m - f) + \frac{2}{3}\epsilon(1 - \bar{b}k_{y0}^2)(f_m - f)^{3/2} \right. \\
 &\quad \left. + \bar{b}\left(\frac{\Delta}{2} - 1\right)(f_m - f)^2 + \bar{b}\epsilon\left(\frac{2}{5}\Delta - \frac{2}{3}\right)(f_m - f)^{5/2} \right]. \quad (A3)
 \end{aligned}$$

Untrapped integral (I_U)

$$\begin{aligned}
 I_U &= \int_{f_m}^{\infty} d\bar{W} JN_U(\bar{W}, k_y), \\
 &= \int_{f_m}^{\infty} d\bar{W} \frac{a}{1+W_*} \left[1 - \bar{b}k_{y0}^2 - \bar{b}(\bar{W} - f) \right] N_{0U} \\
 &\quad \times \left[1 + \left(\frac{\bar{W} - f_m}{\Delta} \right)^2 \right]^{-1}. \quad (A4)
 \end{aligned}$$

Let $\frac{\bar{W} - f_m}{\Delta} = x$ so that $d\bar{W} = \Delta dx$ then

$$I_U = \frac{aN_{0U}\Delta}{1+W_*} \int_0^{\infty} dx \left[1 - \bar{b}k_{y0}^2 - \bar{b}(f_m - f + x\Delta) \right] [1 + x^2]^{-1}, \quad (A5)$$

$$= \frac{aN_{0U}\Delta}{1+W_*} \left[1 - \bar{b}k_{y0}^2 - \bar{b}(f_m - f) \right] \frac{\pi}{2}. \quad (A6)$$

In evaluating the last step, the other diverging integral $\int_0^{\infty} dx x(1+x^2)^{-1}$ has been ignored, as it is a mathematical artifact of the expansion of FLR terms in Eq. (51).

Using the values of I_T and I_U obtained above yield the desired Eq. (50).

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